# Section 4.5

## L'Hôpital's Rule and Indeterminate Forms

(1) Refresher: Determinate and Indeterminate Forms

(2) L'Hôpital's Rule

(3) Comparing Growth of Functions

# The Form of a Limit

The **form** of a limit $\lim_{x\to c} \square$ is the expression resulting from substituting $x = c$ into $\square$.

The form of a limit is **not** the same as its value!
It is a **tool for inspecting** the limit.

$\lim_{x\to 0} x^{\arctan(x)}$ : form $0^0$ $\qquad$ $\lim_{x\to\infty} (1+x)^{\frac{1}{x}}$ : form $\infty^0$

$\lim_{x\to 0} \cos(x)^{\frac{1}{x}}$ : form $1^\infty$ $\qquad$ $\lim_{x\to 0^+} \ln(x)\sin(x)$ : form $0\cdot\infty$

**Indeterminate Forms** are called indeterminate because they represent limits which may or may not exist and may be equal to any value. The form itself does **not** indicate the value of the limit. There are 7 indeterminate forms.

$$\frac{0}{0} \qquad \pm\frac{\infty}{\infty} \qquad \pm 0\cdot\infty$$

$$1^{\infty} \qquad 0^{0} \qquad \infty^{0}$$

$$\infty - \infty$$

# Indeterminate Forms

Limits of form $\dfrac{0}{0}$:

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1 \qquad \lim_{x\to 0} \frac{\sin^2(x)}{x} = 0 \qquad \lim_{x\to 0} \frac{\sin(x)}{x^2} \text{ DNE}$$

Limits of form $\dfrac{\infty}{\infty}$:

$$\lim_{x\to \infty} \frac{x+1}{x} = 1 \qquad \lim_{x\to \infty} \frac{\ln(x)}{x} = 0 \qquad \lim_{x\to \infty} \frac{e^x}{x} = \infty$$

These two indeterminate forms are like "tugs-of-war" between the numerator and denominator. Which of the two grows faster?

# L'Hôpital's Rule

## L'Hôpital's Rule

If $f$ and $g$ are differentiable near $x = a$ and either

(i) $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$, or

(ii) $\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$,

then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

The rule applies equally for one-sided limits.

# Example 1:

These limits are of form $0/0$. The steps marked $\overset{LHR}{=}$ use L'Hôpital's Rule.

$$(a)\ \lim_{x\to 0} \frac{\sin(x)}{x}$$

$$(b)\ \lim_{x\to 0} \frac{\ln(x^2+1)}{x}$$

$$(c)\ \lim_{x\to 2} \frac{x^4+2x-20}{x^3-8}$$

# Example 2:

These limits are of form $\frac{\infty}{\infty}$. The steps marked $\overset{LHR}{=}$ use L'Hôpital's Rule.

$$(a)\ \lim_{x\to\infty} \frac{3x-7}{6x+5}$$

$$(b)\ \lim_{x\to\infty} \frac{\ln(x)}{x}$$

$$(c)\ \lim_{x\to 0^+} \frac{\ln(x)}{\ln(\sin(x))}$$

# Example 3:

Sometimes it is necessary to perform L'Hôpital's Rule multiple times.

$(a)\ \lim_{x \to 0} \frac{e^x - x - 1}{\cos(x) - 1}$ $(0/0)$

$(b)\ \lim_{x \to \infty} \frac{-3x^3 - 20x^2}{2x^3 + x + 7}$ $(\infty/\infty)$

# L'Hôpital's Rule: Warnings

**Warning #1:**
L'Hôpital's Rule only applies to the indeterminate forms $0/0$ and $\infty/\infty$. Before applying L'Hôpital's Rule to a limit, verify that it is of one of those two forms.

**Warning #2:**
Don't confuse L'Hôpital's Rule with the Quotient Rule!
They have completely different uses.

- L'Hôpital's Rule is for evaluating limits.
- The Quotient Rule is for evaluating derivatives.

# The Form $\infty - \infty$

To evaluate limits with the form $\infty - \infty$, use algebra to rewrite the expression as a quotient, often in $0/0$ or $\infty/\infty$ form. Often this means finding a common denominator.

**Example 4:** To evaluate $\displaystyle\lim_{x\to 0^+}\left(\frac{1}{x} - \csc(x)\right)$, which has the form $\infty - \infty$, rewrite it:

$$\frac{1}{x} - \csc(x) \quad = \quad \frac{1}{x} - \frac{1}{\sin(x)} \quad = \quad \frac{\sin(x) - x}{x\sin(x)}$$

This is an $0/0$ form, so we can apply L'Hôpital's Rule:

$$\begin{aligned}\lim_{x\to 0^+}\frac{\sin(x)-x}{x\sin(x)} &\overset{LHR}{=} \lim_{x\to 0^+}\frac{\cos(x)-1}{x\cos(x)+\sin(x)} \quad (0/0)\\ &\overset{LHR}{=} \lim_{x\to 0^+}\frac{-\sin(x)}{-x\sin(x)+2\cos(x)} = \boxed{0}\end{aligned}$$

# The Form $0 \cdot \infty$

Limits with the form $0 \cdot \infty$ can easily be converted to $0/0$ or $\infty/\infty$ form.

If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \infty$, then

$$\lim_{x \to a} \underbrace{f(x)g(x)}_{0\cdot\infty \text{ form}} = \lim_{x \to a} \underbrace{\frac{f(x)}{\left(\frac{1}{g(x)}\right)}}_{0/0 \text{ form}} = \lim_{x \to a} \underbrace{\frac{g(x)}{\left(\frac{1}{f(x)}\right)}}_{\infty/\infty \text{ form}}$$

**Examples (5):**

(a) $\displaystyle\lim_{x \to 0^+} x \ln(x)$

(b) $\displaystyle\lim_{x \to \frac{\pi}{2}^-} \left(x - \frac{\pi}{2}\right) \tan(x)$

# Indeterminate Forms Involving Exponents

For limits of the forms $1^\infty$, $0^0$, or $\infty^0$, the key step is to rewrite

$$\lim_{x\to c} \blacksquare \;=\; \lim_{x\to c} e^{\ln(\blacksquare)} \;=\; \lim_{x\to c} \exp(\ln(\blacksquare)) \;=\; \exp\left(\lim_{x\to c} \ln(\blacksquare)\right).$$

- The second equality is valid by the Limit Laws (see §2.4) because $e^x$ is continuous everywhere.
- The "exp" notation avoids humongous expressions in superscript.

This technique changes the limit into one of form $0 \cdot \infty$:

$$\begin{array}{ccccccc}
1^\infty & \to & e^{\ln|1^\infty|} & \to & e^{\infty\cdot\ln(1)} & \searrow & \\
0^0 & \to & e^{\ln|0^0|} & \to & e^{0\cdot\ln(0)} & \to & e^{0\cdot\infty} \\
\infty^0 & \to & e^{\ln|\infty^0|} & \to & e^{0\cdot\ln(\infty)} & \nearrow &
\end{array}$$

# Indeterminate Forms Involving Exponents

**Example 6:** $\lim_{x \to 0^+} x^x$ has the indeterminate form $0^0$.

# Indeterminate Forms Involving Exponents

**Example 7:** $\lim_{x\to 0^+}(-\ln(x))^x$ has the indeterminate form $\infty^0$.

# Indeterminate Forms Involving Exponents

**Example 8:** Let $a$ be any real number, so that $\lim_{x\to 0}(1+ax)^{1/x}$ has the indeterminate form $1^{\infty}$.

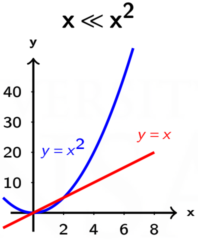
# Comparing Growth Of Functions

We say that $f$ grows faster than $g$ if

(i) $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$, or equivalently

(ii) $\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0$.

(Notation: $g \ll f$.)




Some important cases (that can all be checked using L'Hôpital's Rule):

(a) $x^n \ll e^x$ for all $n$.

(b) In fact, $x^n \ll a^x$ for all $n$ and all $a > 1$.

(c) $\log_a(x) \ll x^n$ for all $n$ and all $a > 0$.

# Indeterminate Forms Involving Trigonometric Functions

**Example 9:** $\lim_{x\to 0} \dfrac{\tan(x)\sin(5x)}{x\sin(7x)}$ has the indeterminate form $\dfrac{0}{0}$.